

# Upper bounds to error probabilities of coded systems beyond the cutoff rate

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## Abstract

*A family of upper bounds to error probabilities of coded systems on the additive white Gaussian noise channel was recently proposed by Divsalar [7]. Their calculation depends only on the weight spectrum of the code words. We first elaborate upon these bounds to show how they can be further tightened by using numerical integration instead of a Chernoff bound, and by reducing the number of code words to be included in the bound. Next, we extend them to fading channels.*

## 1 Introduction and motivation of the work

During the years, much effort has been spent in the search for close approximations to the error probability of systems in which coding is used in conjunction with maximum-likelihood decoding (here we are especially interested in linear binary codes, so we shall restrict our attention to these without any further stipulation). In many cases, the union bound provides a useful tool for the prediction of system performance at intermediate-to-high signal-to-noise ratios (SNR). This is easy to compute, and requires only the knowledge of the weight spectrum of the code; however, it becomes too loose, and hence useless, when the SNR approaches the value at which the cutoff rate  $R_0$  of the channel equals the code rate  $R_c$ .

The recent discovery of an easily decodable family of codes with good error properties even beyond the channel cutoff rate, and close to capacity [6, 8], has rekindled the interest in bounds that overcome the  $R_0$ -limitation of the union bound, while keeping the upsides of it. Specifically, these new bounds should be easily computed, and should depend only on the weight spectrum of the code: the latter property is especially important in view of the fact that with turbo codes only the weights, averaged with respect to the possible choices of the

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interleaver, are usually available. For recent work in this area, see, for example, [9, 10, 14, 15, 16, 17, 18, 20, 21, 22].

Of late, a new family of upper bounds was proposed in [7]. This family turns out to yield the tightest known approximation to the error probability of turbo codes with large block length.

In this manuscript we elaborate on this family of bounds, by showing how it can be further tightened through numerical integration (to be used instead of Chernoff bounding) and through riddance of a number of unnecessary terms. Moreover, we extend these bounds to the fading channel. The paper is organized as follows. In Section 2 we expound the new bounds for the additive Gaussian noise (AWGN) channel, and its improvements. In Section 3 we derive bounds for the fading channel, while in Section 4 some examples of application are shown.

## 2 AWGN channel bounds

Consider transmission of a geometrically-uniform signal constellation  $\mathcal{X}$ , with  $|\mathcal{X}| = M$ , over the AWGN channel, modeled in the form

$$\mathbf{y} = \gamma \mathbf{x} + \mathbf{n} \quad (1)$$

where  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\mathbf{n}$  are  $n$ -dimensional real vectors; in particular,  $\mathbf{x} \in \mathcal{X}$  denotes the transmitted signal vector,  $\mathbf{y}$  the received vector,  $\mathbf{n}$  a random noise vector whose components are Gaussian random variables with mean zero and common variance 1, and  $\gamma$  is a known constant. We also assume that the code word components take on values  $\pm 1$ , so that all signal vectors have equal energy

$$\|\mathbf{x}\|^2 = n \quad (2)$$

With maximum-likelihood (ML) decoding, the word error probability when  $\mathbf{x}$  was transmitted does not depend on  $\mathbf{x}$  due to our assumption of a geometrically-uniform constellation. It can be written in the form

$$P(e) = \mathbb{P} \left[ \bigcup_{\hat{\mathbf{x}} \neq \mathbf{x}} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \right] \quad (3)$$

where  $\{\mathbf{x} \rightarrow \hat{\mathbf{x}}\}$  denote the "pairwise error event," i.e., the probability that when  $\mathbf{x}$  is transmitted the distance between the received vector  $\mathbf{y}$  and  $\hat{\mathbf{x}}$  is smaller than the distance between  $\mathbf{y}$  and  $\mathbf{x}$ , that is,

$$\{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \triangleq \{\mathbf{y} : \|\mathbf{y} - \hat{\mathbf{x}}\| < \|\mathbf{y} - \mathbf{x}\|\} \quad (4)$$

The union in (3) is extended to all signals vectors  $\hat{\mathbf{x}} \neq \mathbf{x}$ . Notice that the set of  $\mathbf{y}$  such that  $\{\mathbf{x} \rightarrow \hat{\mathbf{x}}\}$  occurs is a half-space in  $\mathbb{R}^n$ , the locus of the points whose distance from  $\mathbf{x}$  equals the distance from  $\hat{\mathbf{x}}$ .

The number of terms in the union of (3) can be reduced if we remove all the pairwise error events that can be written as unions of other events. In particular, the minimum number of terms occurs if we keep only the signal vectors  $\mathbf{x}$  that are defined as follows.

Define the Voronoi region of  $\mathbf{x}$  as the set of vectors in the Euclidean  $n$ -dimensional space  $\mathbb{R}^n$  that are closest to  $\mathbf{x}$  than to any other  $\hat{\mathbf{x}}$ , that is

$$\mathcal{V} \triangleq \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \hat{\mathbf{x}}\|, \forall \hat{\mathbf{x}} \in \mathcal{X}\} \quad (5)$$

The Voronoi region is a convex polytope in  $\mathbb{R}^n$ , the intersection of the half-spaces described above. The number of facets in this polytope is usually much lower than  $M$ : for this reason it is convenient to remove the redundancy from (5) and redefine  $\mathcal{V}$  by using only the inequalities that are strictly necessary. To do this, we define the set of (Voronoi) *neighbors* of  $\mathbf{x}$  as the minimal set  $\mathcal{N}$  such that

$$\mathcal{V} = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \hat{\mathbf{x}}\|, \forall \hat{\mathbf{x}} \in \mathcal{N}\} \quad (6)$$

The vector  $\mathbf{x}$  itself does not belong to  $\mathcal{N}$ .

Based on this definition of  $\mathcal{N}$ , we can rewrite the error probability in the form

$$P(e) = \mathbb{P} \left[ \bigcup_{\hat{\mathbf{x}} \in \mathcal{N}} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \right] \quad (7)$$

Now, the union includes only  $|\mathcal{N}|$  terms, the minimum possible number.

For future reference, we can derive from (7) a “minimal” union bound by writing

$$P(e) \leq \sum_{\hat{\mathbf{x}} \in \mathcal{N}} \mathbb{P}\{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \quad (8)$$

A bound tighter than (8) can be obtained as follows. Let  $d$  denote the generic Euclidean distance of  $\hat{\mathbf{x}}$  from  $\mathbf{x}$ , and partition  $\mathcal{N}$  into equivalence classes of vectors with the same value of  $d$ . Denote these by  $\mathcal{N}_d$ , and write

$$\begin{aligned} P(e) &= \mathbb{P} \left[ \bigcup_{d \in \mathcal{D}} \bigcup_{\hat{\mathbf{x}} \in \mathcal{N}_d} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \right] \\ &\leq \sum_{d \in \mathcal{D}} \mathbb{P} \left[ \bigcup_{\hat{\mathbf{x}} \in \mathcal{N}_d} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \right] \\ &= \sum_{d \in \mathcal{D}} \mathbb{P}[e_d] \end{aligned} \quad (9)$$

where  $\mathcal{D}$  is the set of distances from  $\mathbf{x}$  of the vectors in  $\mathcal{X}$ , and

$$e_d \triangleq \bigcup_{\hat{\mathbf{x}} \in \mathcal{N}_d} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} \quad (10)$$

is the probability that, when  $\mathbf{x}$  is transmitted, at least one  $\hat{\mathbf{x}}$  at distance  $d$  is nearer to  $\mathbf{y}$  than  $\mathbf{x}$ .

## 2.1 Characterizing $\mathcal{N}$

The characterization of  $\mathcal{N}$  is a well-studied problem (see, e.g., [1, 2, 3, 13]). Although it generally requires knowledge of the code structure beyond its weight spectrum, useful bounds are available. Let  $\mathcal{X}$  be a linear block code with components 0, 1 and parameters  $n, k$ . If  $w$  denotes the weight of the code word  $\hat{\mathbf{x}}$ , and  $w_{\min}$  the minimum Hamming weight of the code, then the following theorem [2] yields simple bounds to  $\mathcal{N}$ . Here we let  $\mathbf{x}$  be the all-zero code word, so that the Euclidean distance from  $\mathbf{x}$  of a code word with Hamming weight  $w$  is  $2\sqrt{w}$ .

**Theorem.** For any binary linear block code

$$\{\hat{\mathbf{x}} \in \mathcal{X} : 1 \leq w \leq 2d_{\min} - 1\} \subseteq \mathcal{N} \subseteq \{\hat{\mathbf{x}} \in \mathcal{X} : 1 \leq w \leq n - k + 1\} \quad (11)$$

Moreover, if the weight  $w \neq 0$  of  $\hat{\mathbf{x}}$  cannot be written as  $w = i + j$ , where  $i \geq 1, j \geq 1$ , and  $i, j$  are actual weights of words of  $\mathcal{X}$ , then  $\hat{\mathbf{x}} \in \mathcal{N}$ .

For example, as an immediate consequence of (11), for a linear block code we can rewrite (9) as

$$P(e) \leq \sum_{d=2}^{2\sqrt{n-k+1}} \mathbb{P}[e_d] \quad (12)$$

The bound above can be further tightened by using the rest of the Theorem. An algorithm is also available [1] to derive the elements of  $\mathcal{N}$  if the code words can be listed. For example, from tables in [1] we can infer that  $\mathcal{N}$  for the (31, 21) BCH code contains 107,198 words, while  $|\mathcal{X}| = 2,097,152$ . The number of words expurgated from the computation of the bound may consequently be very large, especially when the rate of the code is greater than 1/2 (see [2]).

We should also observe that the words excluded have large distances (in fact, from the Theorem we see that all the words not in  $\mathcal{N}$  have a Hamming weight  $w \geq 2d_{\min}$ ): thus, the effect of this expurgation would be especially felt at low signal-to-noise ratios.

## 2.2 The new bound

We now compute an upper bound to  $P(e)$  based on (9). To do this, use a technique advocated by Gallager in [11] and express  $\mathbb{P}[e_d]$  as

$$\mathbb{P}[e_d] = \mathbb{P}[e_d, \mathbf{y} \in \mathcal{R}] + \mathbb{P}[e_d, \mathbf{y} \notin \mathcal{R}] \quad (13)$$

Further, observe that

$$\mathbb{P}[e_d, \mathbf{y} \notin \mathcal{R}] \leq \mathbb{P}[\mathbf{y} \notin \mathcal{R}] \quad (14)$$

where for the RHS to be a good approximation of the LHS one should choose  $\mathcal{R}$  in such a way that the two regions  $\bar{\mathcal{R}}$  (the complement of  $\mathcal{R}$  in  $\mathbb{R}^n$ ) and  $\bigcup_{\hat{\mathbf{x}} \in \mathcal{N}_d} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\}$  have about the same shape and size. In practice, the selection of  $\mathcal{R}$  should be guided by computational simplicity. The new bound is based on the choice for  $\mathcal{R}$  of an  $n$ -dimensional hypersphere centered at  $\gamma\eta\mathbf{x}$  and with radius  $\sqrt{n}R$ . The parameters  $\eta$  and  $R$  will be selected so as to obtain the tightest possible bound (notice that in general  $\eta$  and  $R$  will depend on  $d$ ).

By using (14) in (13) we obtain the upper bound

$$\mathbb{P}[e_d] \leq \mathbb{P}[e_d, \mathbf{y} \in \mathcal{R}] + \mathbb{P}[\mathbf{y} \notin \mathcal{R}] \quad (15)$$

### 2.2.1 Computation of $\mathbb{P}[\mathbf{y} \notin \mathcal{R}]$

Let us compute the second term in the RHS of (15) first. We have

$$\begin{aligned} \mathbb{P}[\mathbf{y} \notin \mathcal{R}] &= \mathbb{P}\left[\sum_{k=1}^n (y_k - \gamma\eta x_k)^2 \geq nR^2\right] \\ &= \mathbb{P}[W \leq 0] \end{aligned} \quad (16)$$

where we have defined the RV

$$W \triangleq nR^2 - \sum_{k=1}^n (y_k - \gamma\eta x_k)^2 \quad (17)$$

To calculate (16), or to approximate it, we first determine the function

$$\Phi(s) \triangleq \mathbb{E}[e^{sW}] \quad (18)$$

By observing that, under the assumption that  $\mathbf{x}$  was transmitted, we have  $\mathbf{y} = \gamma\mathbf{x} + \mathbf{n}$ , that the components of  $\mathbf{n}$  are independent RVs with mean zero and variance 1, and that  $\|\mathbf{x}\|^2 = n$ , we obtain

$$\Phi(s) = e^{snR^2} \prod_{k=1}^n \mathbb{E}\left[e^{-s((1-\eta)\gamma x_k + n_k)^2}\right]$$

$$\begin{aligned}
 &= e^{snR^2} \prod_{k=1}^n \frac{1}{\sqrt{2s-1}} e^{-(1-\eta)^2 \gamma^2 x_k^2 s / (2s-1)} \\
 &= e^{snR^2} g^n(\gamma, s, \eta)
 \end{aligned} \tag{19}$$

where

$$g(\gamma, s, \eta) \triangleq \frac{1}{\sqrt{2s-1}} e^{-(1-\eta)^2 \gamma^2 s / (2s-1)} \tag{20}$$

This converges in a vertical strip  $\alpha_1 < \alpha < \alpha_2$  of the complex  $s$ -plane bounded by the closest poles of  $\Phi(s)$ .

The exact value of (16) and an upper bound can be determined as described in Appendix A.

### 2.2.2 Computation of $\mathbb{P}[e_d, \mathbf{y} \in \mathcal{R}]$

We have

$$\begin{aligned}
 \mathbb{P}[e_d, \mathbf{y} \in \mathcal{R}] &= \mathbb{P} \left[ \bigcup_{\hat{\mathbf{x}} \in \mathcal{N}_d} \{\mathbf{x} \rightarrow \hat{\mathbf{x}}\}, \mathbf{y} \in \mathcal{R} \right] \\
 &\leq \sum_{\hat{\mathbf{x}} \in \mathcal{N}_d} \mathbb{P}[\{\mathbf{x} \rightarrow \hat{\mathbf{x}}\}, \mathbf{y} \in \mathcal{R}] \\
 &= \sum_{\hat{\mathbf{x}} \in \mathcal{N}_d} \mathbb{P}[\|\mathbf{y} - \hat{\mathbf{x}}\| \leq \|\mathbf{y} - \mathbf{x}\|, \mathbf{y} \in \mathcal{R}] \\
 &= \sum_{\hat{\mathbf{x}} \in \mathcal{N}_d} \mathbb{P}[(\mathbf{y}, \mathbf{x}) \leq (\mathbf{y}, \hat{\mathbf{x}}), \|\mathbf{y} - \gamma\eta\mathbf{x}\|^2 \leq nR^2]
 \end{aligned} \tag{21}$$

The corresponding Chernoff bound takes the form

$$\mathbb{P}[e_d, \mathbf{y} \in \mathcal{R}] \leq A_d e^{-nrR^2} f^d(\gamma, r, \eta) g^{n-d}(\gamma, r, \eta) \tag{22}$$

where  $g(\cdot, \cdot, \cdot)$  was defined in (20), and

$$f(\gamma, s, \eta) \triangleq \frac{1}{\sqrt{2s-1}} e^{-\frac{\gamma^2}{2}(1-2s\eta^2)} \tag{23}$$

Moreover,  $A_d$  is the number of code words at Hamming distance  $d$  from  $\mathbf{x}$ . (For turbo codes, it will denote the average number of code words over all possible interleavers.)

If we select the value of  $e^{nR^2}$  that minimizes the Chernoff bound, and we define for notational simplicity

$$A(\gamma, r, \eta) \triangleq f^d(\gamma, r, \eta) g^{n-d}(\gamma, r, \eta) \tag{24}$$

$$B(\gamma, s, \eta) \triangleq g^n(\gamma, s, \eta) \tag{25}$$

we obtain the neat bound

$$\mathbb{P}[e_d] \leq e^{H(\frac{s}{s-r})} A^{\frac{s}{s-r}}(\gamma, s, \eta) B^{-\frac{r}{s-r}}(\gamma, r, \eta) \quad (26)$$

where  $H(\cdot)$  is the binary entropy function:

$$H(x) \triangleq -x \ln x - (1-x) \ln(1-x). \quad (27)$$

Some algebra is required to minimize the bound (26) with respect to  $s$ ,  $r$ , and  $\eta$ . We obtain

$$\mathbb{P}[e_d] \leq \exp\{H(\rho) - nE(\gamma^2/2, d/n, \rho)\} \quad (28)$$

where

$$E(\gamma^2/2, d/n, \rho) = -\frac{1}{2} \ln(1 - \rho + \rho e^{2(\ln A_d)/n}) + \frac{1}{1 + (1 + \frac{1-\rho}{\rho} e^{-2(\ln A_d)/n}) \frac{1-d/n}{d/n}} \frac{\gamma^2}{2} \quad (29)$$

$$\rho = \frac{1}{1 + \frac{1-\beta}{\beta} e^{2(\ln A_d)/n}} \quad (30)$$

and

$$\beta = \sqrt{\frac{\gamma^2}{2} \frac{1-d/n}{d/n} \frac{2}{1 - e^{-2(\ln A_d)/n}} + \left(\frac{1-d/n}{d/n}\right)^2 \left[ \left(1 + \frac{\gamma^2}{2}\right)^2 - 1 \right] - \left(1 + \frac{\gamma^2}{2}\right) \frac{1-d/n}{d/n}} \quad (31)$$

A discussion of the relations between this new bound, the union bound, and other previously derived bound can be found in [7]. Using modified Gallager bound [7] the factor  $e^{H(\rho)}$  in the bound can be ignored. This bound can be used for bit error probability if  $A_d$  is replaced by  $\sum_w \frac{w}{k} A_{w,d}$  in the bound [7].

### 2.3 Tightening the bound by $Q(\cdot)$ function

Any improved union type bound over AWGN channel can be tightened by replacing  $\mathbb{P}[e_d]$  with  $\min\{\mathbb{P}[e_d], A_d Q(\sqrt{\gamma^2 d})\}$ . The same applies to upper bound on the bit error probability by replacing  $A_d$  with  $\sum_w \frac{w}{k} A_{w,d}$ .

## 3 Fading channel bounds

Here we assume a frequency-flat, slow-fading channel. To obtain an appropriate mathematical model for it, we must consider two factors, viz., the coherence time of the physical channel and

the presence of a delay constraint. The combination of these two factors dictates the choice of the model.

Consider first the coherence time; this is the inverse of the Doppler spread. The product of the coherence time  $T_c$  and the data rate (in symbols per second) yields the number  $L$  of transmitted symbols that are affected approximately by the same fading gain. As an order of magnitude, for many wireless systems the values of the coherence time range from 0.01 to 1 s, while the data rates range from 20 to 200 k symbols per second. Consequently,  $L \geq 20,000 \times 0.01 = 200$  symbols. If the transmitted code word has length  $n$ , we may obtain that for each symbol to be affected by an independent fading gain we should use an interleaver spanning at least  $nL$  symbols. Hence, the actual delay involved may become very large (on the order of  $nT_c$ ). Now, in some applications large delays are unacceptable (for example, real-time speech requires a delay not exceeding 100 ms). Consequently, in the presence of delay constraints in the system an  $n$ -symbol code word will be affected by less than  $n$  independent fading gains. In the following we shall consider separately the two limiting cases of no delay constraint (and hence fading affecting independently every transmitted symbol: the "independent fading channel") and stringent delay constraint (and hence fading affecting independently every transmitted code word: the "block fading channel"). In both cases we assume that the receiver has perfect knowledge of the fading gain affecting the transmission, also known as "channel-state information."

### 3.1 Block fading channel

The channel model here is described by the equation

$$\mathbf{y} = \gamma \alpha \mathbf{x} + \mathbf{n} \quad (32)$$

where  $\alpha$  is the fading gain, a random variable (RV) which we shall assume to be Rayleigh-distributed with  $\mathbb{E}[\alpha^2] = 1$ , i.e., to have the probability density function

$$f_\alpha(r) = 2r e^{-r^2}, \quad r \geq 0 \quad (33)$$

By observing that (32) differs from (1) only for the presence of the random gain  $\alpha$ , if  $p(\gamma)$  denotes the error probability for a given signal constellation over the AWGN channel with parameter  $\gamma$ , then the error probability for the same constellation transmitted over the block fading channel is given by

$$P(e) = \mathbb{E}_\alpha[p(\alpha \gamma)] \quad (34)$$

where  $\mathbb{E}_\alpha[\cdot]$  denotes expectation with respect to the RV  $\alpha$ . In particular, any bound on  $p(\gamma)$  is transformed into a bound for the block fading channel by taking its expectation after the transformation  $\gamma \rightarrow \alpha \gamma$ . The expectation can be evaluated numerically for example by using Laguerre quadrature formulas.

### 3.2 Independent fading channel

The model here is

$$\mathbf{y} = \gamma \boldsymbol{\alpha} \mathbf{x} + \mathbf{n} \quad (35)$$

where  $\boldsymbol{\alpha} = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$  is the diagonal matrix of the fading gains affecting the components of the transmitted vector  $\mathbf{x}$ . We use here the upper bound, derived from (15),

$$\mathbb{P}[e_d] = \mathbb{E}_{\boldsymbol{\alpha}} \mathbb{P}[e_d | \boldsymbol{\alpha}] \leq \mathbb{E}_{\boldsymbol{\alpha}} \mathbb{P}[e_d, \mathbf{y} \in \mathcal{R} | \boldsymbol{\alpha}] + \mathbb{E}_{\boldsymbol{\alpha}} \mathbb{P}[\mathbf{y} \notin \mathcal{R} | \boldsymbol{\alpha}] \quad (36)$$

#### 3.2.1 Selection of the region $\mathcal{R}$

Notice first that the decision metric in this case is based on the minimization of the norm

$$\|\mathbf{y} - \gamma \boldsymbol{\alpha} \mathbf{x}\| \quad (37)$$

The simplest region suggested by (37) is a sphere with radius  $\sqrt{n}R$  centered at  $\eta \gamma \boldsymbol{\alpha} \mathbf{x}$ , where  $\eta$  and  $R$  are parameters to be optimized:

$$\mathcal{R} = \{\mathbf{y} \mid \|\mathbf{y} - \eta \gamma \boldsymbol{\alpha} \mathbf{x}\|^2 \leq nR^2\} \quad (38)$$

However, the resulting bound is not tight.

Another choice is that of an ellipsoid, obtained by rescaling each coordinate of  $\mathbf{y}$  so as to compensate the effect of fading and centered at  $\eta \gamma \mathbf{x}$ :

$$\mathcal{R} = \{\mathbf{y} \mid \|\boldsymbol{\alpha}^{-1} \mathbf{y} - \eta \gamma \mathbf{x}\|^2 \leq nR^2\} \quad (39)$$

where  $\eta$  and  $R$  have to be optimized. This choice does not seem to lead to feasible analytical computations.

Yet another choice consists of a sphere centered at a point obtained by a linear transformation of  $\gamma \boldsymbol{\alpha} \mathbf{x}$ :

$$\mathcal{R} = \{\mathbf{y} \mid \|\mathbf{y} - \mathbf{A} \gamma \boldsymbol{\alpha} \mathbf{x}\|^2 \leq nR^2\} \quad (40)$$

where  $\mathbf{A}$  and  $R$  are to be optimized. A simple choice for the transformation represented by  $\mathbf{A}$  is a rotation and a rescaling, corresponding to a diagonal  $\mathbf{A}$  all of whose elements on the main diagonal being equal to  $\zeta e^{j\psi}$ :

$$\mathcal{R} = \{\mathbf{y} \mid \|\mathbf{y} - \zeta e^{j\psi} \gamma \boldsymbol{\alpha} \mathbf{x}\|^2 \leq nR^2\} \quad (41)$$

where  $\zeta$ ,  $\psi$ , and  $R$  are now the parameters to be optimized. Notice that in this case we have

$$\|\mathbf{y} - \zeta e^{j\psi} \gamma \boldsymbol{\alpha} \mathbf{x}\|^2 = \|\mathbf{y}\|^2 + \zeta^2 \gamma^2 \|\boldsymbol{\alpha} \mathbf{x}\|^2 - 2\zeta \cos \psi \gamma (\mathbf{y}, \boldsymbol{\alpha} \mathbf{x}) \quad (42)$$

In the following we present the bounds resulting from this choice of  $\mathcal{R}$ .

### 3.2.2 The new bound

By replicating the computations described in the previous section, we obtain again a bound in the form

$$\mathbb{P}[e_d] \leq \exp\{H(\rho) - nE(\gamma^2/2, d/n, \rho, \beta, r, \phi)\} \quad (43)$$

where

$$\begin{aligned} E(\gamma^2/2, d/n, \rho, \beta, r, \phi) \triangleq & -\rho(\ln A_d)/n + \frac{\rho}{2} \ln \frac{\beta}{\rho} + \frac{1-\rho}{2} \ln \frac{1-\beta}{1-\rho} \\ & + \rho \frac{d}{n} \ln \left[ 1 + \frac{\gamma^2}{2} (1 - 2r\phi) \right] \\ & + \rho(1 - d/n) \ln \left[ 1 + \frac{\gamma^2}{2} \left( 1 - 2r\phi - \frac{(1-r)^2 \rho}{\beta} \right) \right] \\ & + (1-\rho) \ln \left[ 1 + \frac{\gamma^2}{2} \left( \frac{1-\rho(1-2r\phi)}{1-\rho} - \frac{(1-\rho(1-r))^2}{(1-\rho)(1-\beta)} \right) \right] \end{aligned} \quad (44)$$

This bound should be minimized with respect to  $\phi$ ,  $\rho$ ,  $\beta$ , and  $r$ . The minimum with respect to  $\phi$  can be obtained in a closed-form, then the remaining minimizations must be performed numerically. The bound can be further tightened by replacing  $\mathbb{P}[e_d]$  with  $\min\{\mathbb{P}[e_d], A_d \frac{1}{\pi} \int_0^{\pi/2} \left[ \frac{\sin^2 \theta}{\sin^2 \theta + \frac{\gamma^2}{2}} \right]^d d\theta\}$ . Also, as it will be discussed in the next section, the factor  $e^{H(\rho)}$  can be ignored.

The suboptimum choice  $\rho = \beta = 1$ ,  $r = 0$ , and  $\phi = 0.5$  yields

$$E(\gamma^2/2, d/n, 1, 1, 0, 0.5) = -\frac{\ln A_d}{n} + \frac{d}{n} \ln \left[ 1 + \frac{\gamma^2}{2} \right], \quad (45)$$

which is the exponent of the union bound. Thus, the union bound becomes useless when this exponent equals zero, that is, when

$$\ln \left[ 1 + \frac{\gamma^2}{2} \right] = \max_{d/n} \frac{(\ln A_d)/n}{d/n} \quad (46)$$

For large block length  $n$  and random codes we have

$$\frac{\ln A_d}{n} = H(d/n) - (1 - R_c) \ln 2 \quad (47)$$

so that

$$\max_{d/n} \frac{(\ln A_d)/n}{d/n} = -\ln [2^{1-R_c} - 1] \quad (48)$$

and the minimum SNR for the validity of the bound turns out to be

$$\left(\frac{\gamma^2}{2}\right)_{\min} = \frac{1}{2^{1-R_c} - 1} - 1 \quad (49)$$

Since  $\gamma^2 = 2R_c \mathcal{E}_b/N_0$ , this corresponds to

$$\frac{\mathcal{E}_b}{N_0} = \frac{1}{R_c(2^{1-R_c} - 1)} - \frac{1}{R_c} \quad (50)$$

which is the cutoff rate of the independent Rayleigh fading channel.

Numerical calculations show that for  $R_c = 1/2$  the new bound can predict the error probability above 3.06 dB, while the cutoff rate is 4.515 dB. Thus, the new bound outperforms the union bound by 1.5 dB (it should also be noticed that the capacity for a rate-1/2 code is 1.8 dB). When  $R_c \rightarrow 0$ , the union-bound validity threshold is 1.46 dB, while the new bound is valid up to -1 dB. This indicates that the new bound is tighter for low-rate codes.

## 4 Union type bounds for modified Gallager bounds

For an  $(n, k)$  block code  $\mathcal{C}$ , with  $\mathbf{x}'_m \in \mathcal{C}$ , let's divide the codewords  $\{\mathbf{x}_{m'}\}$  with Hamming distance  $d$  from transmitted codeword  $\mathbf{x}_m$  into subsets  $\chi_d$ ,  $d = 0, 1, 2, \dots, n$ . Cardinality of these sets are  $|\chi_d| = A_d$ , where  $A_d$  is the number of codewords at Hamming distance  $d$  from  $\mathbf{x}_m$ . In the following when the random variables are continuous the  $\sum$  should be replaced by  $\int$ . Gallager in [12] proposed a second bounding technique on word error probability given by

$$P_w \leq \sum_d \sum_y P(\mathbf{y}|\mathbf{x}_m) \left\{ \sum_{\mathbf{x}'_m \in \chi_d: m' \neq m} \left[ \frac{P(\mathbf{y}|\mathbf{x}'_m)}{P(\mathbf{y}|\mathbf{x}_m)} \right]^\lambda \right\}^\rho \quad (51)$$

This bound was further upperbounded by Duman and Salehi [10] as follows: Define a non-negative function  $f(\mathbf{y})$  that represent a density function i.e.  $\sum_y f(\mathbf{y}) = 1$ . Then for parameter  $0 \leq \rho \leq 1$  the above bound, using Jensen's inequality, can be upper bounded as

$$P_w \leq \sum_d \left\{ \sum_{\mathbf{x}'_m \in \chi_d: m' \neq m} \sum_y f(\mathbf{y}) \left[ \frac{f(\mathbf{y})}{P(\mathbf{y}|\mathbf{x}_m)} \right]^{-\frac{1}{\rho}} \left[ \frac{P(\mathbf{y}|\mathbf{x}'_m)}{P(\mathbf{y}|\mathbf{x}_m)} \right]^\lambda \right\}^\rho \quad (52)$$

Sason and Shamai [19] using calculus of variation and iterative method similar to that proposed by Gallager [11] obtained an optimum  $f(y) = \prod_j f(y_j)$  (with no symmetry constraint on  $f(y)$ ) for Duman and Salehi upperbound. The bound is then applied to independent Rician fading channels with channel-state-information (CSI).

Based on the bounding technique by Duman and Salehi in (52), the following bounds in [7] were obtained for any nonnegative function  $f(y)$ ,  $0 \leq \rho \leq 1$ ,  $s = -r\rho/(1 - \rho) \geq 0$ ,  $r \leq 0$ ,  $\lambda \geq 0$

$$\begin{aligned} P_w(\mathbf{x}_m) &\leq \sum_d \sum_y P(y|\mathbf{x}_m) \left\{ \sum_{\mathbf{x}'_m \in \chi_d: m' \neq m} \left[ \frac{P(y|\mathbf{x}'_m)}{P(y|\mathbf{x}_m)} \right]^\lambda \right\}^\rho \\ &\leq \sum_d \left\{ \left[ \sum_y P(y|\mathbf{x}_m) \left[ \frac{f(y)}{P(y|\mathbf{x}_m)} \right]^s \right]^{(1-\rho)} \right. \\ &\quad \times \left. \left\{ \sum_{\mathbf{x}'_m \in \chi_d: m' \neq m} \sum_y P(y|\mathbf{x}_m) \left[ \frac{f(y)}{P(y|\mathbf{x}_m)} \right]^r \left[ \frac{P(y|\mathbf{x}'_m)}{P(y|\mathbf{x}_m)} \right]^\lambda \right\}^\rho \right\} \end{aligned} \quad (53)$$

and a bound on the bit error probability was obtained [7] using method of indicator functions and Jensen's inequality.

$$\begin{aligned} P_b(\mathbf{x}_m) &\leq \sum_d \left\{ \left[ \sum_y P(y|\mathbf{x}_m) \left[ \frac{f(y)}{P(y|\mathbf{x}_m)} \right]^s \right]^{(1-\rho)} \right. \\ &\quad \times \left. \left\{ \sum_{\mathbf{x}'_m \in \chi_d: m' \neq m} \frac{d_H(\mathbf{u}'_m, \mathbf{u}_m)}{k} \sum_y P(y|\mathbf{x}_m) \left[ \frac{f(y)}{P(y|\mathbf{x}_m)} \right]^r \left[ \frac{P(y|\mathbf{x}'_m)}{P(y|\mathbf{x}_m)} \right]^\lambda \right\}^\rho \right\} \end{aligned} \quad (54)$$

The upper bound on word error probability except for factor  $e^{H(\rho)}$  is similar to the results by Gallager in [11]. These results establish the relation between the Gallager bounds in [12] and in [11]. Furthermore the region  $\mathcal{R}$  can be defined as

$$\mathcal{R} = \{y \mid \ln \left[ \frac{f(y)}{P(y|\mathbf{x}_m)} \right] \leq nR\} \quad (55)$$

This provides a geometric interpretation for the Gallager bound. Using this geometric interpretation, in [7] the following have been shown. The region for Viterbi Viterbi bound [21] is the same as in the tangential bound of Berlekamp [5]. The region for Duman Salehi bound [10] is the same as our region, thus a closed form solution to Duman Salehi bound was obtained. For modified Gallager bound, using a cone region as in the tangential sphere bound of Poltyerv [14] and Sason Shamai [15], a closed form solution to simplified version of tangential sphere bound was obtained. The exponents of these closed form bounds are identical to the exponent

of our simple bound. Thus asymptotically as block size  $n \rightarrow \infty$  the simple bound should be as tight as the tangential sphere bound. For more details see [7].

Note that for  $s = 1$ , and  $\sum_{\mathbf{y}} f(\mathbf{y}) = 1$  the modified Gallager bound reduces to Duman Salehi bound (see [19]).

For binary input symmetric output discrete memoryless channels assuming  $f(\mathbf{y}) = \prod_j f(y_j)$ , and  $f(y) = f(-y)$ <sup>1</sup> using calculus of variation as in Gallager [11], and minimizing the upper-bound with respect to  $f(y)$  the following results were obtained [7].

$$P_w \leq \sum_d e^{-n E(c,d)} \quad (56)$$

where

$$E(c, d) = \max_{\rho, \lambda, \beta} \{ -\rho[r(\delta) + \delta \ln h(\rho, \lambda, \beta) + (1 - \delta) \ln g(\rho, \lambda, \beta)] - (1 - \rho) \ln [g(\rho, \lambda, \beta) + \beta h(\rho, \lambda, \beta)] \} \quad (57)$$

and for bit error probability

$$P_b \leq \sum_d e^{-n E_b(c,d)} \quad (58)$$

where

$$E_b(c, d) = \max_{\rho, \lambda, \beta} \{ -\rho[r_b(\delta) + \delta \ln h(\rho, \lambda, \beta) + (1 - \delta) \ln g(\rho, \lambda, \beta)] - (1 - \rho) \ln [g(\rho, \lambda, \beta) + \beta h(\rho, \lambda, \beta)] \} \quad (59)$$

where  $\delta = d/n$ ,  $r(\delta) = \frac{\ln A_d}{n}$ ,  $r_b(\delta) = \frac{\ln \sum_w \frac{w}{k} A_{w,d}}{n}$ ,  $\lambda \geq \frac{1}{2}$ ,  $\frac{\delta}{1-\delta} \leq \beta \leq 1$ ,  $0 \leq \rho \leq 1$ . For random codes it can be shown that  $\beta = 1$ , and  $\lambda = \frac{1}{1+\rho}$  minimizes the bound, and the minimum SNR threshold coincides with capacity limit for random codes.  $\beta$  is the solution to the nonlinear equation

$$\beta = \frac{\delta}{1 - \delta} \frac{g(\rho, \lambda, \beta)}{h(\rho, \lambda, \beta)} \quad (60)$$

which depends on values of  $\rho$ ,  $\lambda$ , and  $\delta$ . Rather than solving this nonlinear equation, for every  $\rho$ ,  $\lambda$ , and  $\delta$ ,  $\beta$  may be treated as a parameter to be optimized numerically to obtain maximum exponent for each  $\delta$ .

$$g(\rho, \lambda, \beta) = \sum_{y>0} P(y|x=0) \left[ \frac{1 + P(y)^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta P(y)^\lambda + P(y)^{2\lambda}} \right]^{1-\rho} [1 + P(y)^{2\lambda}] \quad (61)$$

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<sup>1</sup>with no symmetry assumption, except for normalization of  $f(y)$ , the bound reduces to Duman Salehi-Sason Shamai bound in [19]

$$h(\rho, \lambda, \beta) = \sum_{y>0} P(y|x=0) \left[ \frac{1 + P(y)^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta P(y)^\lambda + P(y)^{2\lambda}} \right]^{1-\rho} 2P(y)^\lambda \quad (62)$$

where  $P(y) \triangleq \frac{P(y|x=1)}{P(y|x=0)}$ , and  $P(y|x)$  is the channel transition probability (likelihood function).

#### 4.1 AWGN channel

For binary input AWGN channel  $P(y) = e^{-2y\sqrt{2c}}$ ,  $P(y|x=0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sqrt{2c})^2}$ ,  $c = \frac{RcE_b}{N_o}$ , and  $\sum_y$  should be replaced by  $\int$ . Let  $z = e^{-2y\sqrt{2c}}$ , then we have

$$g(\rho, \lambda, \beta) = \int_0^1 \frac{e^{-[\frac{(\ln z + 4c)^2}{16c} + \ln z]}}{\sqrt{16c\pi}} \left[ \frac{1 + z^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta z^\lambda + z^{2\lambda}} \right]^{1-\rho} [1 + z^{2\lambda}] dz \quad (63)$$

$$h(\rho, \lambda, \beta) = \int_0^1 \frac{e^{-[\frac{(\ln z + 4c)^2}{16c} + \ln z]}}{\sqrt{16c\pi}} \left[ \frac{1 + z^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta z^\lambda + z^{2\lambda}} \right]^{1-\rho} 2z^\lambda dz \quad (64)$$

#### 4.2 Independent fading channel with CSI

For independent fading channels such as Rician, Rayleigh, Nakagami with density  $p(\alpha)$ ,  $P(y, \alpha) = e^{-2\alpha y\sqrt{2c}}$ ,  $P(y|\alpha, x=0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\alpha\sqrt{2c})^2}$ , and

$$g(\rho, \lambda, \beta) = \int_0^\infty \int_0^\infty p(\alpha) P(y|\alpha, x=0) \left[ \frac{1 + P(y, \alpha)^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta P(y, \alpha)^\lambda + P(y, \alpha)^{2\lambda}} \right]^{1-\rho} [1 + P(y, \alpha)^{2\lambda}] d\alpha dy \quad (65)$$

$$h(\rho, \lambda, \beta) = \int_0^\infty \int_0^\infty p(\alpha) P(y|\alpha, x=0) \left[ \frac{1 + P(y, \alpha)^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta P(y, \alpha)^\lambda + P(y, \alpha)^{2\lambda}} \right]^{1-\rho} 2P(y, \alpha)^\lambda d\alpha dy \quad (66)$$

where fading samples  $\alpha$  are known at the receiver (with CSI), and  $E\{\alpha^2\} = 1$ .

We further simplify these expressions for Rayleigh fading channels as,

$$g(\rho, \lambda, \beta) = \int_0^1 \frac{z^{\frac{1}{2}(\sqrt{1+1/c}-3)}}{4c\sqrt{1+1/c}} \left[ \frac{1 + z^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta z^\lambda + z^{2\lambda}} \right]^{1-\rho} [1 + z^{2\lambda}] dz \quad (67)$$

$$h(\rho, \lambda, \beta) = \int_0^1 \frac{z^{\frac{1}{2}(\sqrt{1+1/c}-3)}}{4c\sqrt{1+1/c}} \left[ \frac{1 + z^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta z^\lambda + z^{2\lambda}} \right]^{1-\rho} 2z^\lambda dz \quad (68)$$

### 4.3 Independent fading channel with no CSI

For independent fading channels with density  $p(\alpha)$ , if the fading samples  $\alpha$  are not provided to the receiver (no CSI), and  $E\{\alpha^2\} = 1$ , then

$$P(y|x = k) = \int_0^\infty p(\alpha) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-(1-2k)\alpha\sqrt{2c})^2} d\alpha, \quad k = 0, 1 \quad (69)$$

and  $P(y) = \frac{P(y|x=1)}{P(y|x=0)}$ . So we can still use the results for AWGN (see the following equations) by replacing the likelihood functions for AWGN with the likelihood functions for independent fading with no CSI.

$$g(\rho, \lambda, \beta) = \int_0^\infty P(y|x = 0) \left[ \frac{1 + P(y)^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta P(y)^\lambda + P(y)^{2\lambda}} \right]^{1-\rho} [1 + P(y)^{2\lambda}] dy \quad (70)$$

$$h(\rho, \lambda, \beta) = \int_0^\infty P(y|x = 0) \left[ \frac{1 + P(y)^{\frac{1-2\rho\lambda}{1-\rho}}}{1 + 2\beta P(y)^\lambda + P(y)^{2\lambda}} \right]^{1-\rho} 2P(y)^\lambda dy \quad (71)$$

For Rayleigh fading case we have

$$P(y|x = k) = \frac{e^{-0.5y^2}}{\sqrt{2\pi}(1+c)} \left[ 1 + (1-2k)\sqrt{2\pi} \frac{\sqrt{c}}{\sqrt{1+c}} y e^{\frac{c}{2(1+c)}y^2} Q\left((2k-1)\frac{\sqrt{c}}{\sqrt{1+c}}y\right) \right], \quad k = 0, 1 \quad (72)$$

### 4.4 Examples

The simple bound for AWGN channel was used to obtain the ML word error probability of rate  $1/2$   $(n, j, k)$  Low Density Parity Check (LDPC) Codes as shown in Fig. 1. In the example rate  $1/2$   $(n, j, k)$  Low Density Parity Check codes for  $n=10000$ , and  $j=3, 4, 5, 6$   $k=2j$  are considered.

In the second example, as shown in Fig. 2, the simple bound for AWGN channel is applied to obtain the ML performance of rate  $1/4$  Repeat Accumulate (RA) codes. Also in the Figure the performance of suboptimum iterative turbo decoder for RA codes are shown.

In the third example, as shown in Fig. 3, the simple bound for Rayleigh fading channel is applied to obtain the ML performance of rate  $1/4$  Repeat Accumulate (RA) codes. Also in the Figure the performance of suboptimum iterative turbo decoder for RA codes over independent Rayleigh fading with CSI are shown.

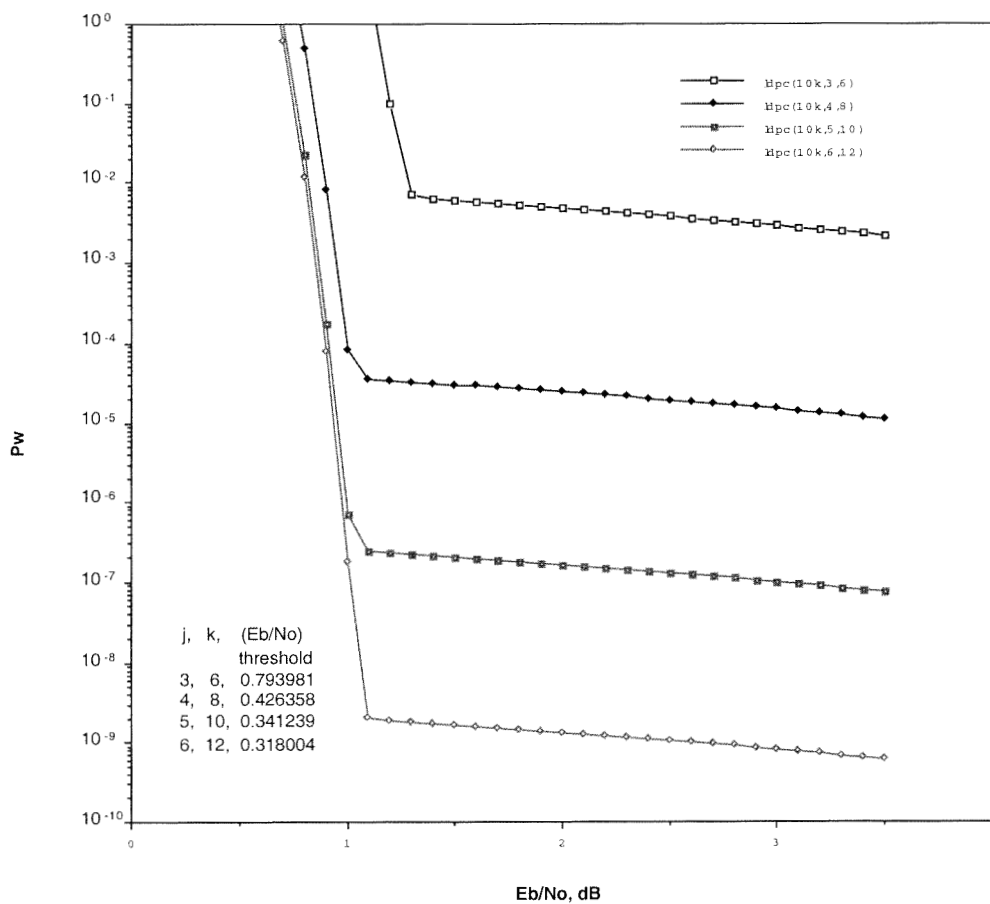


Figure 1: Performance of low density parity check codes using the simple closed form bound for AWGN

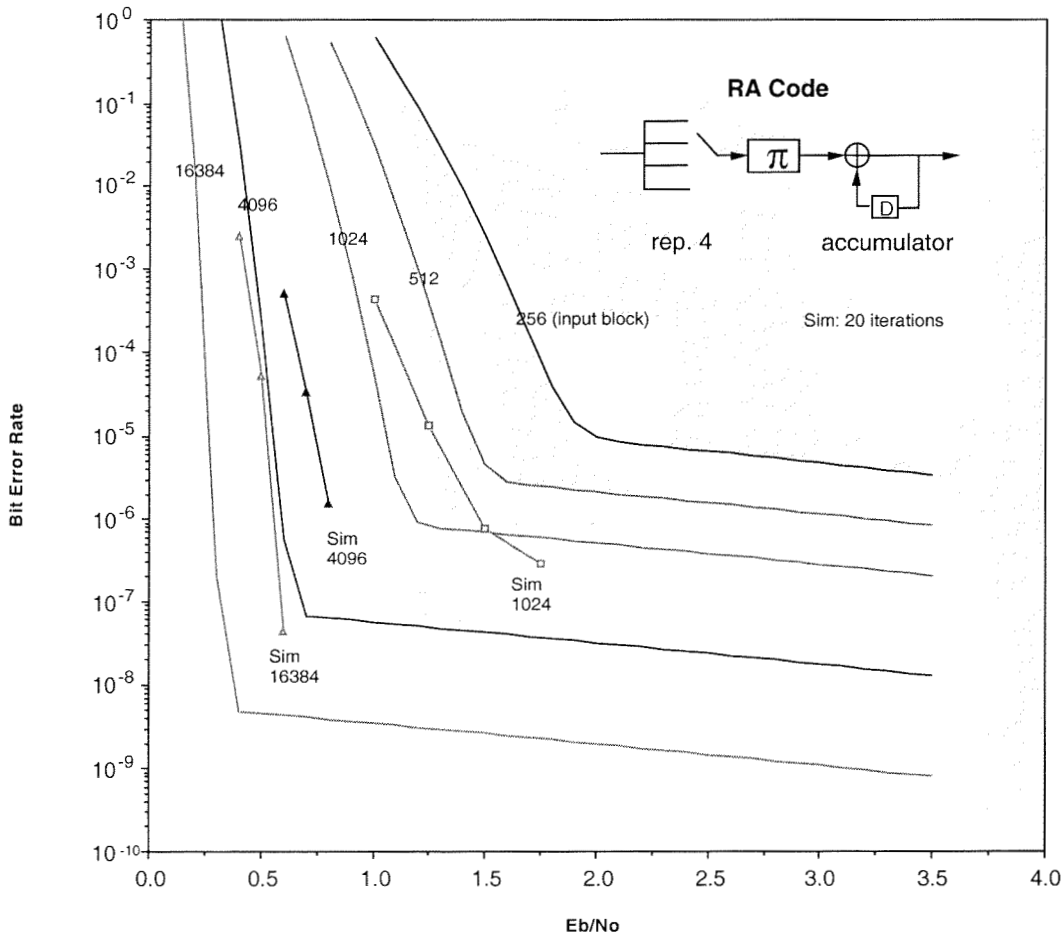


Figure 2: ML upperbound on the bit error probability of rate 1/4 RA codes using the simple closed form bound for AWGN, and the performance of suboptimum iterative turbo decoder

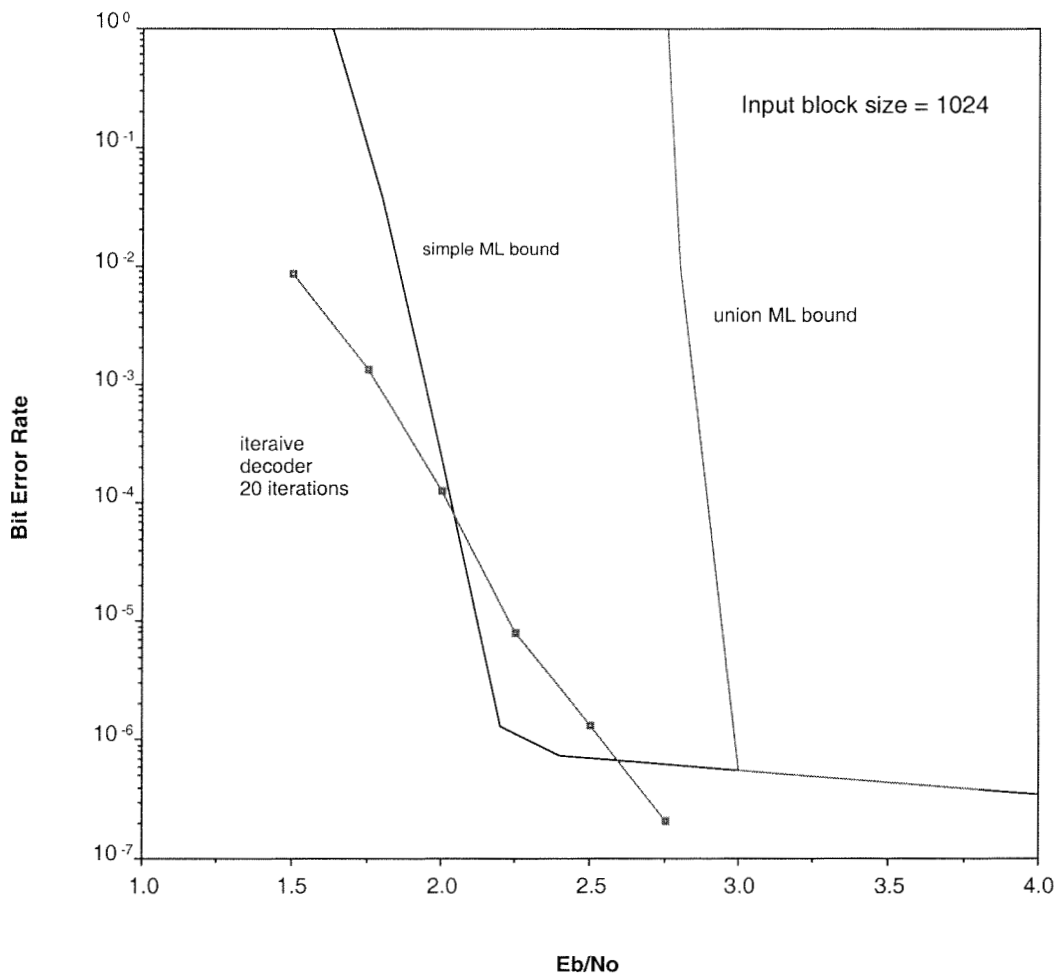


Figure 3: ML upperbound on the bit error probability of rate 1/4 RA codes using the simple bound for Rayleigh fading, and the performance of suboptimum iterative turbo decoder

## A Numerical calculations

### Computing $\mathbb{P}[X \leq 0]$

The computation of  $\mathbb{P}[X \leq 0]$  can be performed as follows. Define first the function

$$\Phi_X(s) \triangleq \mathbb{E}[e^{sX}] \quad (73)$$

This converges in a vertical strip  $\alpha_1 < \alpha < \alpha_2$  bounded by the closest poles of  $\Phi_X(s)$ . Then the Chernoff bound on  $\mathbb{P}[X \leq 0]$  has the form

$$\mathbb{P}[X \leq 0] \leq \min_{0 < \lambda < \alpha_2} \Phi_X(\lambda) \quad (74)$$

for  $\lambda$  real and belonging to the convergence region of (73). The minimum is unique due to the convexity of the restriction of  $\Phi_X(s)$  to the real axis [4, p. 703].

For the exact evaluation of  $\mathbb{P}[X \leq 0]$  we may use the procedure described in [4, pp. 704–705]. Observe first the equality from Laplace-transform theory:

$$\mathbb{P}[X \leq 0] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \Phi_X(s) \frac{ds}{s} \quad (75)$$

To compute numerically the integral above, rewrite (75) as

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(c+j\omega)}{c+j\omega} d\omega &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi_X(c+j\omega)}{c+j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-1}^1 \left(1 - j \frac{\sqrt{1-x^2}}{x}\right) \Phi_X\left(c + jc \frac{\sqrt{1-x^2}}{x}\right) \frac{dx}{\sqrt{1-x^2}} \end{aligned} \quad (76)$$

where the change of variables  $\omega = c\sqrt{1-x^2}/x$  has been made; next, use the Gauss-Chebyshev integration rule

$$\int_{-1}^1 f(x) \frac{dx}{\sqrt{1-x^2}} \approx \frac{\pi}{n} \sum_{i=1}^n f\left(\cos \frac{2i-1}{2n} \pi\right) \quad (77)$$

to obtain

$$\mathbb{P}[X \leq 0] \approx \frac{1}{2n} \sum_{i=1}^n (1 - jx_i) \Phi_X(c \cdot (1 + jx_i)) \quad (78)$$

where

$$x_j \triangleq \tan \frac{2j-1}{2n} \pi \quad (79)$$

Observe that the calculation of (78) could be further simplified by observing that the left-hand side of the equation is real; however, to take advantage of this observation one has to evaluate separately the real and the imaginary part of  $\Phi_X(s)$ , which might not be an easy task.

### Computing $\mathbb{P}[X \leq 0, Y \leq 0]$

Here we have

$$\Phi_{X,Y}(s, t) \triangleq \mathbb{E} [e^{sX+tY}] \quad (80)$$

Then the Chernoff bound on  $\mathbb{P}[X \leq 0, Y \leq 0]$  has the form

$$\mathbb{P}[X \leq 0, Y \leq 0] \leq \min_{\lambda, \mu} \Phi_{X,Y}(\lambda, \mu) \quad (81)$$

for  $\lambda, \mu$  real and belonging to the convergence region of (80).

Exact evaluation of  $\mathbb{P}[X \leq 0, Y \leq 0]$  can be done as follows:

$$\mathbb{P}[X \leq 0, Y \leq 0] = \frac{1}{(2\pi j)^2} \int_{c-j\infty}^{c+j\infty} \int_{d-j\infty}^{d+j\infty} \Phi_{X,Y}(s, t) \frac{ds}{s} \frac{dt}{t} \quad (82)$$

To compute numerically the integral above, rewrite (82) as

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\Phi_{X,Y}(c + j\omega, d + jw)}{(c + j\omega)(d + jw)} d\omega dw \\ &= \frac{1}{4\pi^2} \int_{-1}^1 \int_{-1}^1 \left(1 - j \frac{\sqrt{1-x^2}}{x}\right) \left(1 - j \frac{\sqrt{1-y^2}}{y}\right) \cdot \\ & \quad \cdot \Phi_{X,Y} \left( c + jc \frac{\sqrt{1-x^2}}{x}, d + jd \frac{\sqrt{1-y^2}}{y} \right) \frac{dx}{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-y^2}} \end{aligned} \quad (83)$$

where the change of variables  $\omega = c\sqrt{1-x^2}/x$  and  $w = d\sqrt{1-y^2}/y$  has been made; next, use the Gauss-Chebyshev product-integration rule to obtain

$$\mathbb{P}[X \leq 0, Y \leq 0] \approx \frac{1}{4mn} \sum_{i=1}^n \sum_{k=1}^m (1 - jx_i)(1 - jy_k) \Phi_{X,Y}(c \cdot (1 + jx_i), d \cdot (1 + jy_k)) \quad (84)$$

where

$$x_i \triangleq \tan \frac{2i-1}{2n} \pi, \quad y_k \triangleq \tan \frac{2k-1}{2m} \pi \quad (85)$$

## B Voronoi regions for the fading channel

Here we prove that the reduction of the union bound based on the structure of the Voronoi regions of the code is feasible also when the fading channel (32) is considered. We prove this by showing that the pairwise error probabilities with fading can be written as the probabilities of crossing the same hyperplanes as for the AWGN channel.

Observe first that with channel-state information the maximum-likelihood detection rule is

$$\hat{\mathbf{x}} = \min_{\mathbf{x}}^{-1} \|\mathbf{y} - \alpha \mathbf{x}\| \quad (86)$$

Thus we have, by observing that  $\|\alpha \mathbf{x}\|$  has the same value for all  $\mathbf{x}$  and defining the unit-step function  $u[\cdot]$  and the random vector  $\mathbf{z} \triangleq \alpha \mathbf{z}$ :

$$\begin{aligned} \mathbb{P}\{\mathbf{x} \rightarrow \hat{\mathbf{x}}\} &= \mathbb{E}_{\alpha} \mathbb{P}(\|\mathbf{y} - \alpha \hat{\mathbf{x}}\|^2 < \|\mathbf{y} - \alpha \mathbf{x}\|^2 \mid \alpha) \\ &= \mathbb{E}_{\alpha} \mathbb{P}((\mathbf{y}, \alpha(\hat{\mathbf{x}} - \mathbf{x})) > 0 \mid \alpha) \\ &= \mathbb{E}_{\alpha} \mathbb{P}((\alpha \mathbf{y}, \hat{\mathbf{x}} - \mathbf{x}) > 0 \mid \alpha) \\ &= \mathbb{E}_{\alpha} \mathbb{E}_{\mathbf{y} \mid \alpha} [u[(\alpha \mathbf{y}, \hat{\mathbf{x}} - \mathbf{x})] \mid \alpha, \mathbf{y}] \\ &= \mathbb{E}_{\mathbf{z}} [u[(\mathbf{z}, \hat{\mathbf{x}} - \mathbf{x})]] \\ &= \mathbb{P}[(\mathbf{z}, \hat{\mathbf{x}} - \mathbf{x}) > 0] \\ &= \mathbb{P}(\|\mathbf{z} - \hat{\mathbf{x}}\|^2 < \|\mathbf{z} - \mathbf{x}\|^2) \end{aligned} \quad (87)$$

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